EQUATIONS AND TRANPOSITION OF FORMULAE

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1. Introduction

Most of the Mathematics of Engineering/Science consists of relationships between various physical quantities. These are expressed as Mathematical equations.

Examples

(i) Area of a circle, radius \( r \)

\[ A = \pi r^2 \]

(ii) Coulombs Law. The force of attraction between two charged particles is

\[ F = \frac{k Q_1 Q_2}{r^2} \]

where \( Q_1, Q_2 \) are the charges, \( r \) is their distance apart, \( k \) is a physical constant.

(iii) Rate of heat transfer \( Q \) through a slab of heat conducting material, thickness \( \ell \) is

\[ Q = \frac{k(T_1 - T_0)}{\ell} \]

where \( T_1 - T_0 \) the is temperature difference, \( k \) is the thermal conductivity.

Equations such as the above which represent frequently used results are known as formulae. When using a formula connecting physical quantities it is of course important to use a consistent set of physical units, but in these notes we are solely interested in the Mathematical rules for manipulating equations.

2. Equations

There is only one rule for changing the appearance of an equation.

Whatever you do to one side of the equation you must do the same to the other side.
In the following \( L \) and \( R \) represent whatever may appear on the left and right sides of an equation respectively and \( k \) represents any numerical or algebraic quantity.

If \( L = R \)

then (i) \( R = L \)

**Example**

If \( x + y = z \)

then \( z = x + y \)

The equation may be written either way.

Rather than write \( 2 = x \) we would write \( x = 2 \). Rather than write \( 0 = x^2 + 2x + 3 \) we would write \( x^2 + 2x + 3 = 0 \).

(ii) \( L + k = R + k \)

**Example**

If \( y - z = x^2 \)

then \( y - z + z = x^2 + z \)

i.e. \( y = x^2 + z \)

If you move a negative quantity from one side to the other it becomes positive.

(iii) \( L - k = R - k \)

**Example**

If \( y + z = x^2 \)

then \( y + z - z = x^2 - z \)

i.e. \( y = x^2 - z \)

If you move a positive quantity from one side to the other it becomes negative.

(iv) \( kL = kR \)

**Examples**

(a) If \( \frac{y}{z} = x^2 + 1 \)
\[ z \times \frac{y}{z} = z(x^2 + 1) \]
\[ \text{i.e. } y = z(x^2 + 1) \]

(b) If \[ z\left(\frac{y}{z} + 1\right) = zx^2 \]
\[ y + z = zx^2 \]

This is sometimes called ‘multiplying through’ by \( z \).

(v) \[ \frac{L}{k} = \frac{R}{k} \]

Example

If \[ z(x + y) = 4 \]
then \[ \frac{z(x + y)}{z} = \frac{4}{z} \]
\[ \therefore x + y = \frac{4}{z} \]

This is called ‘dividing through’ by \( z \).

(vii) \( (L)^p = (R)^p \) where \( p \) is a rational number

Examples

(i) If \( \sqrt{x + y} = z \)
then \( (\sqrt{x + y})^2 = z^2 \)
\[ \therefore x + y = z^2 \]

(ii) If \( (x + y)^2 = z \)
then \( \sqrt{(x + y)^2} = \sqrt{z} \)
i.e. \[ x + y = \sqrt{z} \]

The following set of examples are concerned with solving equations by making use of the above rules.

Examples
(i) \[ x - 2 = 4 \]
\[ \therefore x = 4 + 2 = 6 \] (Add 2 to each side)

(ii) \[ x + 2 = 4 \]
\[ \therefore x = 4 - 2 = 2 \] (Subtract 2 from each side)

(iii) \[ \frac{x}{2} + 1 = 4 \]
\[ \therefore \frac{x}{2} = 3 \] (Multiply through by 2)
\[ \therefore x = 2 \times 3 = 6 \]

(iv) \[ 4 + 4x = 16 \]
\[ \therefore 4x = 12 \] (Divide through by 2)
\[ \therefore x = \frac{12}{4} + 3 \]

(v) \[ \frac{4}{x} - 1 = 5 \]
\[ \therefore \frac{4}{x} = 6 \] (Invert both sides)
\[ \therefore \frac{x}{4} = \frac{1}{6} \]
\[ \therefore x = \frac{4}{6} = \frac{2}{3} \]

All the above are examples of LINEAR EQUATIONS. Any equation which can be written in the form \[ ax + b = 0 \] where \( a \) and \( b \) are real numbers is called a LINEAR EQUATION. The value of \( x \) which satisfies the equation is called the root of the equation. Linear equations have one root \( x = -\frac{b}{a} \).

(vi) \[ \sqrt{x^2 + 9} = 5 \] (Square both sides)
\[ \therefore x^2 + 9 = 25 \]
\[ \therefore x^2 = 16 \] (Square root of both sides)
\[ \therefore x = \pm 4 \]

(vii) \[ \frac{x}{2} = \frac{2}{x} \] (Multiply through by \( x \))
\[ \frac{x^2}{2} = 2 \] (Multiply through by 2)
\[ x^2 = 4 \]
\[ x = \pm 2 \]
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(ix) \[ \frac{3}{x} + x = 4 \]  
\[ 3 + x^2 = 4x \] (Multiply through by x)  
\[ x^2 - 4x + 3 = 0 \]  
\[ (x - 3)(x - 1) = 0 \]  
\[ x = 3 \quad \text{or} \quad x = 1 \]

(x) \[ 3x^2 = x \]  
It is very tempting to divide through by x BUT DON’T  
\[ 3x^2 - x = 0 \]  
\[ x \] is a common factor  
\[ x(3x - 1) = 0 \]  
\[ x = 0 \quad \text{or} \quad x = \frac{1}{3} \]  
If we had divided through by x the root at x = 0 would have been missed.

Examples (vi)-(x) are examples of QUADRATIC EQUATIONS. These are covered in detail in the notes FACTORISATION AND QUADRATIC EQUATIONS. As we saw they may have two real roots, one repeated root, or no real roots.

(xi) \[ x^3 - 2x^2 = 3x \]  
It is very tempting to divide through by x BUT DON’T  
\[ x^3 - 2x^2 - 3x = 0 \]  
\[ x \] is a common factor  
\[ x(x^2 - 2x - 3) = 0 \]  
The expression in brackets factorises  
\[ x(x - 3)(x + 1) = 0 \]  
\[ x = 0 \quad \text{or} \quad x = 3 \quad \text{or} \quad x = -1 \]  
Example (xi) is a CUBIC EQUATION. The general form is \[ ax^3 + bx^2 + cx + d = 0 \] where \( a, b, c \) and \( d \) real numbers. A cubic equation may have three real roots or only one real root. There is a formula for solving cubic equations but it is very complicated and not much used instead we tend to use computational methods which will be covered later in your course.

Tutorial 1

In exercises 1-10 find the values of \( x \) satisfying the equations.

1. \[ x + 6 = 5 \]  
2. \[ 2x + 7 = 9 \]  
3. \[ \frac{x}{5} + 1 = \frac{x}{10} \]  
4. \[ \frac{x}{2} = \frac{8}{x} \]  
5. \[ \frac{x + 1}{x + 2} = 2 \]  
6. \[ \frac{x + 1}{x + 2} = \frac{x + 3}{x - 6} \]  
7. \[ \frac{1}{2x - 3} = \frac{3}{x + 5} \]  
8. \[ x + 1 = \frac{(x + 3)(x - 2) + 6}{x} \]  
9. \[ \sqrt{x^2 + 1} = 2 \]  
10. \[ (x + 1)^\frac{2}{3} + (x + 1)^\frac{1}{3} = 0 \]
11. \( \frac{3}{2x} = x + \frac{3}{2} \cdot \frac{1}{x} \)

12. \( \sqrt{x^3 + 1} = 3x + 1 \)

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3. Transposition of Formulae

An important application of the above rules occurs in what is called ‘transposition of formulae’.

We all know the formula for the area of a circle: \( A = \pi r^2 \)

Hence it is easy to calculate the area of a circle of radius 0.5m.

\[ A = \pi (0.5)^2 = 0.7854\text{m}^2 \]

But what if we are told that the area of a circle is 2 m\(^2\) and we need to know the radius?

We need to rewrite the formula as \( r = \sqrt{\frac{A}{\pi}} \). This is called transposing the formula to make \( r \) the subject of the formula.

It is now quite easy to calculate the required radius:

\[ r = \sqrt{\frac{2}{\pi}} = 0.7979\text{m} \]

In the following examples the aim is to make the given variable the subject of the formula.

1. \( F = \frac{kQ_1Q_2}{r^2} \) \( \text{(r)} \)

Rewrite as \( r^2F = kQ_1Q_2 \). Then \( r^2 = \frac{kQ_1Q_2}{F} \) and finally \( r = \sqrt{\frac{kQ_1Q_2}{F}} \).

2. \( Q = \frac{k(T_1 - T_0)}{l} \) \( \text{(T)} \)

First \( Ql = k(T_1 - T_0) \) then \( \frac{Ql}{k} = T_1 - T_0 \) and finally \( T_1 = T_0 + \frac{Ql}{k} \)

3. \( a = \sqrt{b^2 + c^2} \) \( \text{(b)} \)

First \( a^2 = b^2 + c^2 \) then \( b^2 = a^2 - c^2 \) and finally \( b = \sqrt{a^2 - c^2} \)

Let us look at this in more detail. In the given formula the following operations are carried out on the variable \( b \) :-

(i) Square it
(ii) Add \( c^2 \)
(iii) Take square root

To make \( b \) the subject we need to retrace these steps. So we need to carry out the inverse operations in reverse order on the subject of the formula as given which is \( a \).

Inverse of (iii) gives \( a^2 \)
Inverse of (ii) gives \( a^2 - c^2 \)
Inverse of (i) gives \( \sqrt{a^2 - c^2} \)

This technique is used in the next example.
3. \[ S = 4\pi r \sqrt{\frac{R^2 + r^2}{2}} \quad (R) \]

Operations on R :-
(i) Square
(ii) Add \( r^2 \)
(iii) Divide by 2
(iv) Square root
(v) Multiply by \( 4\pi r \)

Inverse operations in reverse order (to be performed on \( S \)) .

(i) Divide by \( 4\pi r \) \quad \text{Gives} \quad \frac{S}{4\pi r}
(ii) Square \quad \text{Gives} \quad \frac{S^2}{16\pi^2 r^2}
(iii) Multiply by 2 \quad \text{Gives} \quad \frac{S^2}{8\pi^2 r^2}
(iv) Subtract \( r^2 \) \quad \text{Gives} \quad \frac{S^2}{8\pi^2 r^2 - r^2}
(v) Square root \quad \text{Gives} \quad \sqrt{\frac{S^2}{8\pi^2 r^2 - r^2}}

Therefore the transposed formula is \[ R = \sqrt{\frac{S^2}{8\pi^2 r^2 - r^2}} \]

It needs to be emphasised that the method shown above is of limited application as shown by the next example .

4. \[ R = \frac{R_1 R_2}{R_1 + R_2} \quad (R_i) \]

Before applying the above we would have to rewrite the formula so that the required subject variable only appeared once. In this case it is quicker and easier to proceed as follows:

First \( RR_1 + RR_2 = R_1 R_2 \) then \( RR_2 = R_1 R_2 - RR_1 = R_1 (R_2 - R) \) finally \( R_i = \frac{RR_2}{R_2 - R} \)

5. \[ q = A_1 \sqrt{\frac{2gh}{\left(\frac{A_1}{A_2}\right)^2 - 1}} \quad (A_2) \]

This is a difficult problem . We will do it two ways .

First :-

Divide both sides by \( A_1 \).

\[ \frac{q}{A_1} = \sqrt{\frac{2gh}{\left(\frac{A_1}{A_2}\right)^2 - 1}} \]
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Square both sides

\[ \frac{q^2}{A_2^2} = \frac{2gh}{(A_1/A_2)^2 - 1} \]

Invert both sides

\[ \frac{A_1^2}{q^2} = \frac{(A_1/A_2)^2 - 1}{2gh} \]

Multiply by \( 2gh \)

\[ \frac{2ghA_1^2}{q^2} = \left( \frac{A_1}{A_2} \right)^2 - 1 \]

\[ \frac{2ghA_1^2}{q^2} + 1 = \left( \frac{A_1}{A_2} \right)^2 \]

\[ \frac{2ghA_1^2 + q^2}{q^2} = \left( \frac{A_1}{A_2} \right)^2 \]

Invert both sides

\[ \frac{q^2}{2ghA_1^2 + q^2} = \frac{A_2^2}{A_1^2} \]

\[ \frac{q^2A_1^2}{2ghA_1^2 + q^2} = A_2^2 \]

And finally

\[ A_2 = \frac{qA_1}{\sqrt{2ghA_1^2 + q^2}} \]

Second way:-  Operations on \( A_2 \) :-

(i) Invert (Raise to power \(-1\)) , (ii) \( \times A_1 \) , (iii) Square,

(iv) \(-1\) , (v) Invert , (vi) \( \times 2gh \) , (vii) Square root (viii) \( \times A_1 \).

Inverse in reverse operating on \( q \) we obtain

(i) \( \frac{q}{A_1} \)  (ii) \( \frac{q^2}{A_1^2} \)  (iii) \( \frac{q^2}{2ghA_1^2} \)  (iv) \( \frac{2ghA_1^2}{q^2} \)

(v) \( \frac{2ghA_1^2}{q^2} + 1 \)  (vi) \( \sqrt{\frac{2ghA_1^2}{q^2} + 1} \)  (vii) \( \frac{\sqrt{\frac{2ghA_1^2}{q^2} + 1}}{A_1} \)  (viii) \( \frac{A_1}{\sqrt{\frac{2ghA_1^2}{q^2} + 1}} \)

Thus

\[ A_2 = \frac{A_1}{\sqrt{\frac{2ghA_1^2}{q^2} + 1}} = \frac{A_1q}{\sqrt{2ghA_1^2 + q^2}} \] as before.
Tutorial 2

1. Given that a sphere has volume $10\text{m}^3$ find its radius.

2. The diagram illustrates a vehicle $P$ which starts from 0 with initial velocity $u\text{ms}^{-1}$, and constant acceleration $a\text{ms}^{-1}$.

The various quantities are related by the formula

$$a = \frac{2(u t - s)}{t^2}$$

Given that $a = 2\text{ms}^{-2}$, $u = 10\text{ms}^{-1}$ find $s$, the distance travelled, after 6 seconds.

3. In each of the following transpose the given formula to make the symbol in brackets the subject of the formula.

(a) $R = \frac{R_1 R_2}{R_1 + R_2}$ \hspace{1cm} (R_1) \\
(b) $d = 2\sqrt{h(2r - h)}$ \hspace{1cm} (r) \\
(c) $V = \frac{\pi h(3R^2 + h^2)}{6}$ \hspace{1cm} (R) \\
(d) $v = \frac{IR}{E - IR}$ \hspace{1cm} (I) \\
(e) $q = A_1 \frac{2gh}{\sqrt{(\frac{A_1}{A_2})^2 - 1}}$ \hspace{1cm} (A_1)

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